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Hamiltonian and quasi-Hamiltonian systems, Nambu–Poisson structures and symmetries

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Abstract
The theory of Hamiltonian and quasi-Hamiltonian systems with respect to Nambu–Poisson structures is studied. It is proved that if a dynamical system is endowed with certain properties related to the theory of symmetries then it can be considered as a quasi-Hamiltonian (or Hamiltonian) system with respect to an appropriate Nambu–Poisson structure. Several examples of this construction are presented. These examples are related to integrability and also to superintegrability.

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1. Introduction

The use of additional compatible structures plays a relevant role in the geometric approach to dynamical systems by means of vector fields. This is the reason for the interest in the search of such structures. For instance, Hojman proposed in a recent paper [1] a general technique, valid for systems of both ordinary and partial differential equations, for finding an admissible Hamiltonian structure for a given equation of motion using one infinitesimal symmetry transformation and one constant of motion. Such a technique was extended in subsequent papers [2, 3] for dealing with dynamical systems in field theory without using any Lagrangian. For a recent update of Hojman's approach, see [4] and references therein.

The geometric approach to (Hamiltonian and Lagrangian) mechanics started first with the use of symplectic structures but then other more general formalisms, as presymplectic or Poisson structures, were also considered. Nambu proposed in 1973 [5] a generalization of the classical Hamiltonian formalism for the study of a system defined on a three-dimensional phase space with coordinates (x_1, x_2, x_3) by introducing a new class of brackets for three functions (f_1, f_2, f_3) given by

$$\{f_1, f_2, f_3\} = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)},$$

where the right-hand side denotes the Jacobian determinant. This multibracket allows us to express the time evolution of a function f by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, h_1, h_2\}.$$

Here h_1 and h_2 are two 'Hamiltonian' or 'Nambu' functions that must necessarily be constants of motion. Shortly after that, a certain number of authors studied [6–8] the relation between Nambu and Hamiltonian mechanics.

Some years later Takhtajan [9] introduced the concept of Nambu–Poisson (or simply Nambu) structure using an axiomatic formulation for the m-bracket operation, and this new approach motivated a series of papers on the same subject (see, e.g., [10–15]). The existence of an interesting relation between Nambu–Poisson manifolds and Leibniz algebroids [16, 17] was also proved. Another generalization was the so-called generalized Poisson brackets [18–20]. A comparison of both concepts was given in [21].

As in Poisson geometry, the existence of a Nambu–Poisson bracket is equivalent to the existence of a skew-symmetric contravariant tensor N of order m satisfying a condition equivalent to the fundamental identity. It has been proved [22–24] that a Nambu–Poisson tensor N of order $m \geqslant 3$ is decomposable; as a consequence a Nambu–Poisson manifold is locally foliated.

Our aim in this paper is to analyse the existence of Nambu–Poisson structures appropriate for the study of a given dynamical system, Γ, and to present a technique for the construction of such a structure when the dynamical system is endowed with certain properties related to the theory of symmetries.

The organization of this paper is as follows: section 2 is devoted to introducing the notation and basic definitions and to discussing some relevant properties of Nambu–Poisson manifolds. The possibility of finding a Nambu–Poisson structure appropriate for making the vector field Γ Hamiltonian (or quasi-Hamiltonian) is analysed in section 3. The idea is that if Γ is endowed with certain properties then this construction can be carried out. Section 4 contains several illustrative examples and, finally, we make in section 5 some final comments.

2. Notation and basic definitions

Let M be a smooth n-dimensional manifold and $C^\infty(M)$ denotes the algebra of differentiable real-valued functions on M. A Nambu–Poisson structure of order m is given by an m-dimensional multivector field, i.e. a $C^\infty(M)$-skew multilinear map

$$N : \bigwedge{}^1(M) \times \overset{m}{\cdots} \times \bigwedge{}^1(M) \to C^\infty(M)$$

which in local coordinates $(x_1, x_2, \ldots, x_n)$ is given by

$$N = n_{i_1\cdots i_m}(x) \frac{\partial}{\partial x_{i_1}} \wedge \frac{\partial}{\partial x_{i_2}} \cdots \wedge \frac{\partial}{\partial x_{i_m}},$$

where summation over repeated indices is understood, which allows us to define the bracket of m functions by

$$\{f_1, f_2, \ldots, f_m\} = N(\mathrm{d}f_1, \mathrm{d}f_2, \ldots, \mathrm{d}f_m),$$

which in local coordinates $(x_1, \ldots, x_n)$ turns out to be

$$\{f_1, f_2, \ldots, f_m\} = n_{i_1 i_2 \ldots i_m} \frac{\partial f_1}{\partial x_{i_1}} \frac{\partial f_2}{\partial x_{i_2}} \cdots \frac{\partial f_m}{\partial x_{i_m}},$$

in such a way that the following conditions are satisfied:

(1) Skew-symmetry: given m functions $f_1, \ldots, f_m$,

$$\{f_1, f_2, \ldots, f_m\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(m)}\},$$

where $\sigma \in S_m$ (symmetric group of m elements) and $\epsilon(\sigma)$ denotes its parity.

(2) Multilinearity: if k_1 and k_2 are real numbers,

$$\{k_1 g_1 + k_2 g_2, f_2, \ldots, f_m\} = k_1\{g_1, f_2, \ldots, f_m\} + k_2\{g_2, f_2, \ldots, f_m\}$$

for any $m+1$ functions $g_1, g_2, f_2, \ldots, f_m$.

(3) Leibniz rule: for any $m+1$ functions $g_1, g_2, f_2, \ldots, f_m$,

$$\{g_1 g_2, f_2, \ldots, f_m\} = g_1\{g_2, f_2, \ldots, f_m\} + \{g_1, f_2, \ldots, f_m\} g_2.$$

(4) Generalized Jacobi identity, usually called Fundamental identity (shortened as FI):

$$\begin{aligned}\{f_1, \ldots, f_{m-1}, \{g_m, \ldots, g_{2m-1}\}\} &= \{\{f_1, \ldots, f_{m-1}, g_m\}, g_{m+1}, \ldots, g_{2m-1}\} + \ldots \\ &\ldots + \{g_m, \ldots, g_{2m-2}, \{f_1, \ldots, f_{m-1}, g_{2m-1}\}\},\end{aligned}$$

for any $2m-1$ functions in M, $f_1, \ldots, f_{m-1}, g_m, \ldots, g_{2m-1}$.

Property 4 that is also known as the 'Takhtajan identity' is considered the appropriate generalization of the Jacobi identity characterizing the standard Poisson bracket. As an example for $m = 3$ and $m = 4$ it reduces to

$$\{f_1, f_2, \{g_3, g_4, g_5\}\} = \{\{f_1, f_2, g_3\}, g_4, g_5\} + \{g_3, \{f_1, f_2, g_4\}, g_5\} + \{g_3, g_4, \{f_1, f_2, g_5\}\},$$

and

$$\begin{aligned}\{f_1, f_2, f_3, \{g_4, g_5, g_6, g_7\}\} &= \{\{f_1, f_2, f_3, g_4\}, g_5, g_6, g_7\} + \{g_4, \{f_1, f_2, f_3, g_5\}, g_6, g_7\} \\ &+ \{g_4, g_5, \{f_1, f_2, f_3, g_6\}, g_7\} + \{g_4, g_5, g_6, \{f_1, f_2, f_3, g_7\}\}.\end{aligned}$$

The 'Takhtajan identity' can be presented in some other equivalent ways. The following property [25, 26] gives an alternative form:

Proposition 1. *A multi-derivation* $\{\cdot, \cdot, \ldots, \cdot\} : C^\infty(M) \times C^\infty(M) \cdots \times C^\infty(M) \longrightarrow C^\infty(M)$ *satisfies the FI if and only if*

(1) The FI is true for the coordinate functions x_i, $i = 1, \ldots, n$.

(2) The following quadratic identities are satisfied

$$\begin{aligned}\sum_{k=1}^{m} [\{g_1, f_1, \ldots, f_{m-2}, f_{m+k-1}\}\{g_2, f_m, \ldots, \widehat{f}_{m+k-1}, \ldots, f_{2m-1}\} \\ + \{g_2, f_1, \ldots, f_{m-2}, f_{m+k-1}\}\{g_1, f_m, \ldots, \widehat{f}_{m+k-1}, \ldots, f_{2m-1}\}] = 0. \quad (1)\end{aligned}$$

for $2m$ *arbitrary functions* $f_1, \ldots, f_{m-2}, f_m, \ldots, f_{2m-1}$ *and* g_1, g_2.

The important point is that a set of $m-1$ functions, $f_1, \ldots, f_{m-1}$, defines a vector field to be denoted by $X_{f_1,\ldots,f_{m-1}}$ by contracting N with $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{m-1}$, i.e. $X_{f_1,\ldots,f_{m-1}} g = \{f_1, \ldots, f_{m-1}, g\}$. Such a vector field satisfies $\mathcal{L}_{X_{f_1,\ldots,f_{m-1}}} N = 0$. Moreover, this property for any $m-1$ functions is equivalent to the FI Actually, $\mathcal{L}_{X_{f_1,\ldots,f_{m-1}}} N = 0$ means that

$$X_{f_1,\ldots,f_{m-1}}\{g_1, \ldots, g_m\} = \{X_{f_1,\ldots,f_{m-1}}(g_1), \ldots, g_m\} + \cdots + \{g_1, \ldots, X_{f_1,\ldots,f_{m-1}}(g_m)\}.$$

More generally, for any $k \leqslant m$ we can define a map

$$N^{\#} : \Gamma\big(\bigwedge{}^k(T^*M)\big) \to \Gamma\big(\bigwedge{}^{m-k}(TM)\big)$$

by contraction of N with each k-form in M.

In this formalism, a function $f \in C^\infty(M)$ is a constant of the motion for the Nambu dynamics represented by the Hamiltonian vector field $X_{f_1,\ldots,f_{m-1}}$ if and only if

$$\{f, f_1, f_2, \ldots, f_{m-1}\} = 0.$$

Note also that, as a consequence of the FI, the Nambu–Poisson bracket of m constants of the motion for a Hamiltonian vector field is a constant of motion too.

The vector field $X_{f_1,\ldots,f_{m-1}}$ is said to be Nambu–Hamiltonian. Note that in such a case the functions $f_1, \ldots, f_{m-1}$ are constants of motion, and this means that only vector fields admitting several constants of motion can be Nambu–Hamiltonian vector fields with respect to a Nambu–Poisson structure.

A vector field Y in M for which there exists a function g such that gY is Nambu–Hamiltonian is said to be quasi-Nambu–Hamiltonian. This means that there exist $m-1$ functions $f_1, \ldots, f_{m-1}$, such that $gY = X_{f_1,\ldots,f_{m-1}}$. The functions f_i defining such a vector field are also constants for the corresponding dynamics because of the skew-symmetry of N.

An interesting particular case is when m is equal to the dimension of the manifold M. For instance, if Q is an n-dimensional manifold and $M = T^*Q$ is endowed with its natural symplectic form ω_0, then the multivector in M which is dual of the $2n$-form $\omega_0 \wedge \overset{n}{\cdots} \wedge \omega_0$ defines a Nambu–Poisson structure (in this case it is the dual of the Liouville structure).

Finally, the FI also implies that [13, 15]

$$\big[X_{f_1,\ldots,f_{m-1}}, X_{f_m,\ldots,f_{2m-2}}\big] = \sum_{i=1}^{m-1} X_{f_m,\ldots,f_{m+k-2},X_{f_1,\ldots,f_{m-1}}(f_{m+k-1}),\ldots,f_{2m-2}}.$$

A remarkable property is that, when m is an even number, the FI holds if and only if the Schouten bracket $[N, N]$ vanishes (see, e.g., [18, 20, 27]). Moreover we recall that the Schouten bracket of two decomposable m-vectors is given by

$$\begin{aligned}[X_1 \wedge \cdots \wedge X_m, Y_1 \wedge \cdots \wedge Y_m] = \sum_{i,j=1}^{m} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \\ \cdots \wedge \widehat{X}_i \wedge \cdots X_m \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge Y_m,\end{aligned} \tag{2}$$

where $\widehat{X}_i$ means that the vector field X_i is omitted and the same for $\widehat{Y}_j$.

3. Construction of a Nambu structure out of symmetries and constants of the motion

Up to now, we first assume the existence of a Nambu structure N on a manifold M and then analyse if a certain vector field X on M is Nambu–Hamiltonian with respect to N. Now we study the inverse problem. We start with a dynamical vector field Γ on a phase space M and prove that if this dynamics is endowed with certain properties related to the theory of symmetries, then it can be considered a (quasi-)Hamiltonian system with respect to a certain Nambu structure.

The following theorem provides a method for the construction of the Nambu structure.

Theorem 1. *Let Γ be a dynamical system on a manifold M. Suppose that*

(1) Γ possesses two commuting infinitesimal symmetries represented by the vector fields X_1 and X_2.

(2) There exist two constants of the motion functions for Γ, h_1 *and* h_2.
(3) The action of the bi-vector $X_1\wedge X_2$ *on the exterior product* $\mathrm{d}h_1 \wedge \mathrm{d}h_2$ *does not vanish.*

Then the 3-vector field $N_{012} = \Gamma\wedge X_1\wedge X_2$ *is a Nambu–Poisson structure on M and the dynamical system* Γ *is 'quasi-Hamiltonian' with respect to* N_{012}*. Moreover, a new Nambu–Poisson structure* J *on M, proportional to* N_{012}*, can be defined so that* Γ *is the Hamiltonian vector field of the functions* h_1 *and* h_2 *with respect to* J.

Proof. The expression

$$N_{012} = \Gamma\wedge X_1\wedge X_2 \tag{3}$$

defines a decomposable 3-vector field on M and the fact that X_1 and X_2 are commuting infinitesimal symmetries of Γ,

$$[X_1, \Gamma] = 0, \qquad [X_2, \Gamma] = 0, \tag{4}$$

implies that the distribution generated by X_1, X_2 and Γ is integrable. Takhtajan proved in [9] that in such a case the multivector satisfies the FI Consequently, N_{012} defines a Nambu–Poisson structure that moreover is invariant under Γ, because

$$\mathcal{L}_\Gamma(\Gamma\wedge X_1\wedge X_2) = \Gamma\wedge[\Gamma, X_1]\wedge X_2 + \Gamma\wedge X_1\wedge[\Gamma, X_2] = 0.$$

The Schouten bracket of N_{012} with itself vanishes, that is $[N_{012}, N_{012}] = 0$, and N_{012} is also a generalized Poisson structure invariant under the dynamics.

Note that condition 1 could be modified to the case in which there exists an integrable two-dimensional distribution invariant under Γ. If this integrable distribution is orientable then the proof is still valid in this more general case. In fact, the existence of a 2-vector V on M which is tangent to the distribution and such that $V(x) \neq 0$ for all $\in M$ leads to the Nambu structure $N = \Gamma \wedge V$.

Denote by $\{x_a;\, a = 1, 2, \ldots, n\}$ a local set of coordinates in the manifold M and suppose the following coordinate expressions for the three vector fields:

$$\Gamma = f^a(x)\frac{\partial}{\partial x_a}, \qquad X_1 = z_1^b(x)\frac{\partial}{\partial x_b}, \qquad X_2 = z_2^c(x)\frac{\partial}{\partial x_c}.$$

Then N_{012} is given by

$$N_{012} = n_{abc}\frac{\partial}{\partial x_a}\wedge\frac{\partial}{\partial x_b}\wedge\frac{\partial}{\partial x_c}, \qquad n_{abc} = \det\begin{vmatrix} f^a & f^b & f^c \\ z_1^a & z_1^b & z_1^c \\ z_2^a & z_2^b & z_2^c \end{vmatrix}.$$

The action of $N_{012}^{\#}$ on the two differentials, $\mathrm{d}h_1$ and $\mathrm{d}h_2$, of the two assumed constants of motion for Γ is

$$N_{012}^{\#}(\mathrm{d}h_1, \mathrm{d}h_2) = h_{12}\Gamma,$$

where the function h_{12} is given by

$$h_{12} = X_1(h_1)X_2(h_2) - X_1(h_2)X_2(h_1).$$

Hence the dynamical vector field Γ is 'quasi-Hamiltonian' with respect to the Nambu–Poisson structure N_{012}. On the other hand, the vanishing of the Lie brackets $[X_i, \Gamma]$ means that the corresponding Lie derivatives, $\mathcal{L}_{X_i}$ and $\mathcal{L}_\Gamma$, also commute

$$\mathcal{L}_{[\Gamma, X_i]} = \mathcal{L}_\Gamma\mathcal{L}_{X_i} - \mathcal{L}_{X_i}\mathcal{L}_\Gamma,$$

and because of this the function h_{12} is a constant of the motion for Γ,

$$\mathcal{L}_\Gamma h_{12} = \mathcal{L}_\Gamma[\mathcal{L}_{X_1}(h_1)\mathcal{L}_{X_2}(h_2) - \mathcal{L}_{X_1}(h_2)\mathcal{L}_{X_2}(h_1)] = 0.$$

Since we have $h_{12} \neq 0$, we can therefore define a new structure J as follows,

$$J = \frac{1}{h_{12}} N_{012},$$

so that J is also a Nambu–Poisson structure and Γ satisfies

$$\Gamma = J^{\#}(\mathrm{d}h_1, \mathrm{d}h_2).$$

Thus Γ is the Hamiltonian vector field, with respect to J, of the functions h_1 and h_2.

We close this section by pointing out the necessity of the third point for the proof of this theorem, since if the 2-form constructed out of the two exterior differentials of the functions h_i would be zero when evaluated on the bi-vector field constructed out of the two symmetries, the statement of the theorem would be invalid. For instance, suppose a system Γ that admits separability and can be decoupled in two subsystems Γ_α and Γ_β depending on coordinates $\{x_\alpha; \alpha = 1, 2, \ldots, n_\alpha\}$ and $\{x_\beta; \beta = 1, 2, \ldots, n_\beta\}$, $n_\alpha + n_\beta = n$, respectively. Then if X_1 and X_2 are symmetries of the first subsystem Γ_a (depending only of the x_α) and h_1 and h_2 are integrals of motion for the second subsystem Γ_β (depending only of the x_β) the function h_{12} will vanish in a trivial way.

□

4. Four examples

A Nambu–Hamiltonian dynamical system must necessarily possess several constants of the motion. Conversely, the Nambu formalism seems appropriate for the study of those dynamical systems known to possess integrals of motion. Now we consider four examples. The first one is related to integrability and the other three to superintegrability.

4.1. Central potential

Let Γ be the following vector field,

$$\Gamma = \sum y_i \frac{\partial}{\partial x_i} - kF(r) \sum x_i \frac{\partial}{\partial y_i}, \qquad r^2 = \sum x_i^2,$$

defined on a $2n$-dimensional phase space M with coordinates $\{x_i, y_i; i = 1, 2, \ldots, n\}$ and let us denote by J_{ij} and X_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$, the following functions and vector fields

$$J_{ij} = x_i y_j - x_j y_i, \qquad X_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i}.$$

It is clear that the J_{ij} are constants of motion and the X_{ij} are symmetries

$$\Gamma(J_{ij}) = 0 \qquad \text{and} \qquad [\Gamma, X_{ij}] = 0.$$

Let us define a 3-vector field N_{0ab} as follows:

$$N_{0ab} = \Gamma \wedge X_{ar} \wedge X_{bs}, \qquad a \neq b, r \neq s.$$

Then the distribution generated by Γ, X_{ar} and X_{bs} is completely integrable, N_{0ab} satisfies $[N_{0ab}, N_{0ab}] = 0$, and N_{0ab} is a Nambu–Poisson structure. Then if we consider the two functions $h_1 = J_{ab}$ and $h_2 = J_{rs}$ as Hamiltonians we obtain

$$N_{0ab}^{\#}(\mathrm{d}h_1, \mathrm{d}h_2) = h_{12}\Gamma,$$

where the function h_{12} is given by

$$h_{12} = J_{as}^2 - J_{br}^2.$$

Let us denote by J the new 3-vector field defined by

$$J = \frac{1}{h_{12}}\Gamma \wedge X_{ar} \wedge X_{bs},$$

which is also a Nambu–Poisson structure. The function h_{12} is Γ-invariant, that is $\Gamma(h_{12}) = 0$, and consequently the dynamical vector field Γ is the Hamiltonian vector field, with respect to J, of the functions h_1 and h_2:

$$\Gamma = J^{\#}(\mathrm{d}h_1, \mathrm{d}h_2).$$

4.2. An isotropic harmonic oscillator

Consider a six-dimensional phase space M with local coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the dynamical vector field

$$\Gamma = X_1 + X_2 + X_3, \qquad X_i = y_i\frac{\partial}{\partial x_i} - \omega^2 x_i\frac{\partial}{\partial y_i}, \qquad i = 1, 2, 3.$$

It is clear that the vector fields $X_i, i = 1, 2, 3$, are symmetries of the dynamics (that is, $[X_i, \Gamma] = 0$) and commute among themselves (that is, $[X_i, X_j] = 0$). The distribution generated by X_1, X_2 and X_3 is completely integrable and the following 3-vector field N defined by

$$N_{023} = \Gamma \wedge X_2 \wedge X_3$$

that in this case reduces to

$$N_{123} = X_1 \wedge X_2 \wedge X_3$$

is a Nambu–Poisson structure.

On the other hand, the functions $J_{ij} = x_i y_j - x_j y_i$ are Γ-invariant (that is $\Gamma(J_{ij}) = 0$). Hence, if we consider two of them as Hamiltonians, $h_1 = J_{31}$ and $h_2 = J_{12}$, we arrive at the following result,

$$N_{123}^{\#}(\mathrm{d}h_1, \mathrm{d}h_2) = h_{12}\Gamma,$$

where the function h_{12} is given by

$$h_{12} = F_{12}F_{13}, \qquad F_{ij} = y_i y_j + \omega^2 x_i x_j.$$

Let us denote by J the new 3-vector field defined by

$$J = \frac{1}{h_{12}}X_1 \wedge X_2 \wedge X_3.$$

Each of the functions F_{ij} is Γ-invariant, that is $\Gamma(F_{ij}) = 0$, and consequently the dynamical vector field Γ is the Hamiltonian vector field, with respect to J, of the functions h_1 and h_2:

$$\Gamma = J^{\#}(\mathrm{d}h_1, \mathrm{d}h_2).$$

4.3. The Kepler problem

In a similar way, if we remove the points $(0, 0, 0, y_1, y_2, y_3)$ in the preceding phase space, we can consider the following dynamical vector field:

$$\Gamma = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3} + \frac{k}{r^3}\left(x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2} + x_3\frac{\partial}{\partial y_3}\right), \qquad r^2 = x_1^2 + x_2^2 + x_3^2.$$

Let us denote by J_i the functions $J_i = x_j y_k - x_k y_j$ and by X_i and X_{123} the vector fields

$$X_i = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial y_j} \qquad X_{123} = J_1 X_1 + J_2 X_2 + J_3 X_3,$$

where $(i, j, k) = (1, 2, 3)$ or any circular permutation. Then $X_i, i = 1, 2, 3$, and X_{123} are infinitesimal symmetries of Γ (that is, $[X_i, \Gamma] = 0$ and $[X_{123}, \Gamma] = 0$) such that $[X_i, X_{123}] = 0$. We can particularize $X_i = X_2$ and define a 3-vector field N_{023} by

$$N_{023} = \Gamma \wedge X_2 \wedge X_{123}$$

which is a Nambu–Poisson structure invariant under Γ.

Moreover, one can check that the functions

$$h_1 = R_2, \qquad h_2 = R_3, \qquad \text{with} \qquad R_i = \epsilon_{ijl} J_j y_l - k \frac{x_i}{r}$$

are Γ-invariant, i.e. $\Gamma(R_2) = \Gamma(R_3) = 0$. The Hamiltonian vector field defined by the functions R_2 and R_3 with respect to the Nambu–Poisson tensor N_{023} is given by

$$N_{023}^{\#}(\mathrm{d}R_2, \mathrm{d}R_3) = h_{23}\Gamma,$$

where the function h_{23} is given by

$$h_{23} = R_1(J_1 R_3 - R_2 J_3),$$

and is Γ-invariant, i.e. $\Gamma(h_{23}) = 0$.

The 3-vector field

$$J = \frac{1}{h_{23}} \Gamma \wedge X_2 \wedge X_{123}$$

is then a Nambu–Poisson structure as well and the dynamical vector field Γ is the Hamiltonian vector field, with respect to J, of the functions h_2 and h_3:

$$\Gamma = J^{\#}(\mathrm{d}R_2, \mathrm{d}R_2).$$

4.4. *The Calogero–Moser system*

Consider now a six-dimensional phase space M with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the dynamical system in M given by

$$\Gamma = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3} + 2c_0 \left[\left(\frac{1}{x_{21}^3} - \frac{1}{x_{13}^3} \right) \frac{\partial}{\partial y_1} + \left(\frac{1}{x_{32}^3} - \frac{1}{x_{21}^3} \right) \frac{\partial}{\partial y_2} + \left(\frac{1}{x_{13}^3} - \frac{1}{x_{32}^3} \right) \frac{\partial}{\partial y_3} \right],$$

where use has been made of the notation $x_{ij} = x_i - x_j$.

Let N be the multivector

$$N_{023} = \Gamma \wedge X_2 \wedge X_3,$$

where X_2 and X_3 are given by

$$X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

$$X_3 = \left(y_1^2 + V_{21} + V_{13}\right) \frac{\partial}{\partial x_1} + \left(y_2^2 + V_{32} + V_{21}\right) \frac{\partial}{\partial x_2} + \left(y_3^2 + V_{13} + V_{32}\right) \frac{\partial}{\partial x_3} + \Gamma\left(y_1^2 + V_{21} + V_{13}\right) \frac{\partial}{\partial y_1} + \Gamma\left(y_2^2 + V_{32} + V_{21}\right) \frac{\partial}{\partial y_2} + \Gamma\left(y_3^2 + V_{13} + V_{32}\right) \frac{\partial}{\partial y_3}$$

and V_{ij} denotes the function $V_{ij} = c_0 / x_{ij}^2$.

Note that the vector fields X_2 and X_3 commute, that is $[X_2, X_3] = 0$, and the distribution generated by Γ, X_2 and X_3 is completely integrable.

Moser proved [28] that the n-dimensional Calogero system can be presented as a Lax equation and that a fundamental set of constants of the motion is given by

$$I_k = \frac{1}{k}\,\mathrm{tr}\,A^k, \qquad A = A_1 + \mathrm{i}c_0 A_2, \qquad k = 1, 2, \dots, n,$$

where A_1 and A_2 denote the diagonal and non-diagonal matrices

$$A_1 = \mathrm{diagonal}[y_1, y_2, \dots, y_n], \qquad (A_2)_{ij} = \left[(1 - \delta_{ij})\frac{1}{x_{ij}}\right].$$

Wojciechowski proved the super-integrability of this system [29] by showing the existence of an additional family of integrals (see also [30–33]). If we make use of the matrix Q defined by

$$Q = \mathrm{diagonal}[x_1, x_2, \dots, x_n],$$

then the additional constants of the motion can be given as the traces of products of the matrices Q and A [31]. In the particular case we are considering, if we denote by L_{ij} the functions $L_{ij} = x_i y_j - x_j y_i$, the following two functions,

$$\begin{aligned} h_2 &= [\mathrm{tr}(QA)]I_1 - [\mathrm{tr}(Q)](2I_2) \\ &= L_{21}(y_2 - y_1) + L_{32}(y_3 - y_2) + L_{13}(y_1 - y_3) + \text{terms of lower order} \\ h_3 &= [\mathrm{tr}(QA^2)]I_1 - [\mathrm{tr}(Q)](3I_3) \\ &= L_{21}\big(y_2^2 - y_1^2\big) + L_{32}\big(y_3^2 - y_2^2\big) + L_{13}\big(y_1^2 - y_3^2\big) + \text{terms of lower order} \end{aligned}$$

are Γ-invariant, i.e. $\Gamma(h_2) = \Gamma(h_3) = 0$. The action of N_{023} on the 1-forms $\mathrm{d}h_2$ and $\mathrm{d}h_3$ is

$$N_{023}^{\#}(\mathrm{d}h_2, \mathrm{d}h_3) = h_{23}\Gamma,$$

where the function h_{23} is given by

$$h_{23} = (y_2 - y_1)^2(y_3 - y_2)^2(y_1 - y_3)^2(y_1 + y_2 + y_3) + \text{terms of lower order.}$$

Let us denote by J the new 3-vector field defined by

$$J = \frac{1}{h_{23}}\Gamma \wedge X_2 \wedge X_3.$$

The function h_{23} is Γ-invariant, that is $\Gamma(h_{23}) = 0$, the 3-vector field J is also a Nambu–Poisson structure and the dynamical vector field Γ is the Hamiltonian vector field, with respect to J, of the functions h_2 and h_3:

$$\Gamma = J^{\#}(\mathrm{d}h_2, \mathrm{d}h_3).$$

5. Conclusions and outlook

Recent results have shown the possibility of enlarging the idea of Hamiltonian system which becomes a much more general concept and includes, in addition to the classical one, some other more general cases. A given dynamical system previously considered as a non-Hamiltonian one can however be seen, if certain properties are satisfied, as Hamiltonian but with respect to either an alternative (noncanonical) symplectic structure or even a more general (in some cases degenerate) Poisson structure. Moreover, a system can even be bi-Hamiltonian with respect to two rather different geometric structures.

This paper must be considered as part of a programme devoted to studying nonstandard construction of Hamiltonian structures. We assume as a starting point that the dynamical system enjoys certain symmetry properties. If the system to be studied is described by a differential equation, in geometric terms a vector field Γ, the relevant symmetries are Lie symmetries and in the geometric approach the symmetry properties are written in terms of Lie derivatives and Lie brackets. We have analysed here a technique for the construction of Nambu–Poisson structures compatible with Γ and we have related these structures to the theory of quasi-Hamiltonian and Hamiltonian systems, in the framework of Nambu systems, and we have illustrated the usefulness of such a construction with several interesting examples. It is to be remarked that these structures are (in many cases) highly degenerated and this fact suggests the existence of several related questions deserving a deeper study. Another interesting question is the study of the superintegrable systems using the formalism of the Nambu mechanics (see, e.g., [34–36]) that we have already considered in three particular cases (oscillator, Kepler and Calogero–Moser). A maximally superintegrable system has $2n - 1$ integrals $I_k, k = 1, \ldots, 2n - 1$ (including the Hamiltonian itself) that can be considered as new Hamiltonian functions in a Nambu formalism. The relation of superintegrability with the existence of Nambu structures, as well the relation with the existence of multi-Hamiltonian structures (in this case Γ is Hamiltonian with respect to different symplectic structures [37]), is also an interesting matter to be studied.

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